WS 2016/17

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Symplectic Geometry

Homework 3

Exercise 1. (10 points)
Let V be a $2n$ dimensional vector space. Recall the definition of the exterior product:
• A tensor product of a k-linear map α and a m-linear map β is a $(k+m)$ linear map defined by

$$\alpha \otimes \beta(v_1, \dots, v_{m+k}) = \alpha(v_1, \dots, v_k) \cdot \beta(v_{k+1}, \dots, v_{k+m}).$$

• The Alt of a k-linear map γ is defined by

$$Alt(\gamma)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \gamma(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

• The exterior product of a k-linear map α and a m-linear map β is defined by

$$\alpha \wedge \beta = \frac{(k+m)!}{k!m!} Alt(\alpha \otimes \beta).$$

Let ω be a skew symmetric bilinear form on V. Prove that ω is non-degenerate if and only if n-th fold exterior product of ω, i.e. $\omega^n = \omega \wedge \dots \wedge \omega$ is nonzero.

Exercise 2. (10 points)
Let α and β be, respectively, k- and l-forms on a vector space V, and let v be a vector in V. Show that

$$\iota_v(\alpha \wedge \beta) = (\iota_v \alpha) \wedge \beta + (-1)^k \alpha \wedge \iota_v \beta.$$

Exercise 3. (10 points)
Prove that the exterior derivative $d\colon \Omega^*(M) \to \Omega^{*+1}(M)$ on a manifold M satisfies $d^2 = 0$. (Hint: use local description of d and the equality of mixed partial derivatives.)

Exercise 4. (10 points)
Prove Cartan's magic formula:

$$\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d.$$

A good strategy is to follow the steps:

1. Check the formula for 0-forms $\omega \in \Omega^0(M) = C^\infty(M)$.

2. Check that both sides commute with d.

3. Check that both sides are derivations of the algebra $(\Omega^*(M), \wedge)$. For instance, check that

$$\mathcal{L}_X(\omega \wedge \alpha) = (\mathcal{L}_X \omega) \wedge \alpha + \omega \wedge (\mathcal{L}_X \alpha).$$

4. Notice that, if U is the domain of a coordinate chart, then $\Omega^*(U)$ is generated as an algebra by $\Omega^0(U)$ and $d\Omega^0(U)$, i.e. every element in $\Omega^*(U)$ is a linear combination of wedge products in $\Omega^0(U)$ and elements in $d\Omega^0(U)$.

Bonus Exercise. (0 points)
Find expressions for $\iota_X \alpha$ in local coordinates.

Hand in: Thursday November 10th
in the exercise session
in Übungsraum 1, MI